# 7. Applied welfare economics with discrete choice models: implications of theory for empirical specification

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## 1. INTRODUCTION

The apparatus of the Random Utility Model (RUM) first emerged in the early 1960s, with Marschak (1960) and Block and Marschak (1960) translating models originally developed for discriminant analysis in psychophysics (Thurstone, 1927) to the alternative domain of discrete choice analysis in economics. Whilst some researchers were quick to see its practical potential (e.g. McFadden, 1968, 1975), it was not until the late 1970s and early 1980s that RUM was equipped with a reasonably comprehensive theoretical rationale in terms of the economics of consumption. An important tenet of this rationale was the link between discrete choice and welfare, which established a basis for applying RUM to public policy analysis, and paved the way for the plethora of applications which have been witnessed over the last 30 years.

It will be helpful to clarify precisely what we mean by 'discrete choice', since Small and Rosen (1981) – which will be referred to as 'S&R' in the remainder of this chapter – suggest three alternative rationales, as follows. First, commodities may be available in continuous quantities but only a limited number of varieties. Second, goods may be supplied in discrete units of such magnitude that only a small number of those units are typically consumed (in this case, S&R cite the example of travel mode choice). Third, if the search for the optimal consumption bundle entails a choice between alternative corner solutions, then the problem is reduced to discrete units. S&R draw particular motivation from the first rationale, introducing a general model of demand comprising both continuous and discrete components. That is to say, an individual is represented as choosing a quantity of a continuous commodity conditional upon discrete choice.

Whilst not overlooking the significant contributions of McFadden

(1981) and Williams (1977), S&R's analysis has proved particularly influential in establishing a basis upon which discrete choice models can be applied to welfare economics. S&R state that 'The purpose of [their] paper is to demonstrate that the conventional methods of applied welfare economics can be generalised to handle cases in which discrete choices are involved' (S&R, 1981, p. 106). Furthermore, they remark that: 'Throughout, the emphasis is on providing rigorous guidelines for carrying out empirical work' (p. 106). It is notable that, despite the intensity with which RUM has been applied over the last 30 years, S&R's paper has stood the test of time; the key propositions of the paper remain largely unchallenged and continue to underpin the analysis of significant public policy interventions.

That said, in the years following its publication, a small but significant literature (e.g. Hau, 1985, 1987; Jara-Díaz and Farah, 1988; Jara-Díaz, 1990; Karlström, 1999; Karlström and Morey, 2004) has clarified the properties of the consumer surplus measure emanating from S&R's paper. In particular, these contributors have considered the extent to which S&R admits income effects of both price and income changes. The present paper will seek to contribute to the aforementioned literature by furthering understanding of S&R, especially in a manner that appeals to the aspiration to provide '. . .rigorous guidelines for carrying out empirical work'. More specifically, our chapter will offer four substantive contributions, as follows:

1.  Section 2 will introduce S&R's problem of discrete-continuous demand, before articulating the concept of a probabilistic demand function, and exposing the assumptions underlying this concept.
2.  Section 3 will consider the application of the Slutsky equation to the discrete and continuous components of demand, from both individual-level and aggregate perspectives. In this regard, the present chapter will present a definitive account of the assumptions underpinning S&R's derivation of the Slutsky equation.
3.  Sections 4 and 5 will reconcile S&R's model of discrete-continuous demand with four fundamental properties of demand functions, namely 'adding up', 'negativity', 'homogeneity' and 'symmetry'. For a restricted case of S&R's model involving only the probabilistic demand, the chapter will identify particular requirements on model specification, such that the aforementioned properties hold.
4.  Finally, section 6 will review the rationale followed by S&R in deriving consumer surplus from discrete choice models. It will be shown that the 'log sum' measure of consumer surplus implies very particular requirements on model specification, consistent with those supporting the fundamental properties of demand functions.

## 2. A MODEL OF DISCRETE-CONTINUOUS DEMAND

### 2.1 Introduction

This section will summarise in formal terms S&R's model of discrete-continuous demand. We shall first consider individual-level demand which, strictly speaking, is the relevant perspective for the fundamental properties of demand functions. We shall then proceed to consider the implications for aggregate demand, which is the perspective more commonly adopted by discrete choice analysts. That is to say, analysts usually interpret the probabilistic demand function associated with discrete choice as deriving from inter-individual rather than intra-individual variation in preferences.

### 2.2 Discrete-continuous demand, in principle

Following S&R, consider a maximisation problem wherein the individual consumes non-negative quantities of three goods. We shall assume that goods 1 and 2 are mutually exclusive, whilst the third good (which we refer to as good $n$) acts as a 'numeraire'. Defining notation: $u$ is direct utility; $\mathbf{x} = (x_1, x_2, x_n)$ is a vector comprising the quantities of goods 1, 2 and the numeraire good; $\mathbf{p} = (p_1, p_2, 1)$ is the associated vector of prices of goods 1, 2 and $n$ (noting that the price of the numeraire good is normalised to one); and $y$ is total income. Adopting this notation, we can write the maximisation problem, thus:

$$
\begin{aligned}
\text{Max} \quad & u = u(\mathbf{x}) \\
\text{s.t.} \quad & \mathbf{px} = y \\
& x_1 x_2 = 0 \\
& \mathbf{x} \geq 0
\end{aligned}
\tag{7.1}
$$

Since the budget constraint within (7.1) implies an ability to straightforwardly exchange between income $y$ and the numeraire good $x_n$, we can restate the problem entirely equivalently:

$$
\begin{aligned}
\text{Max} \quad & u = u(\mathbf{x}) \\
\text{s.t.} \quad & p_1 x_1 + p_2 x_2 = y_{12} \\
& x_1 x_2 = 0 \\
& \mathbf{x} \geq 0
\end{aligned}
$$

where $y_{12} = y - x_n$

This restatement of the problem alludes to the potential for combining good 1 or 2 with good $n$ to form composite goods.[1] Note that, in the restricted case where all income is devoted to goods 1 and 2, it must hold that $x_n = 0$ and $y_{12} = y$; this case will be a recurring point of discussion in what follows.

The constraint $x_1 x_2 = 0$, which precludes simultaneous consumption of goods 1 and 2, is especially pertinent to the present chapter, since it embodies S&R's notion of discrete-continuous demand. More specifically, S&R conceptualise a two-stage consumption decision. In the first stage, the individual chooses between goods 1 and 2 according to which yields the greater utility:

$$u^*(\mathbf{x}) = v^*(\mathbf{p}, y) = \tilde{v}_k(p_k, y) = \text{Max}\{\tilde{v}_1(p_1, y), \tilde{v}_2(p_2, y)\} \quad (7.2)$$

where $u^*$ is the maximum direct utility given income $y$, $v^*$ is the corresponding maximum indirect utility, $\tilde{v}_k$ is the conditional indirect utility (i.e. conditional upon choice), and $k$ indexes the chosen (i.e. utility maximising) good, i.e. $k = 1$ if $\tilde{v}_1 \geq \tilde{v}_2$, or $k = 2$ otherwise.[2] Having chosen between goods 1 and 2, the individual selects a positive quantity of only the chosen good (i.e. consistent with $x_1 x_2 = 0$).

In solving (7.1), S&R employ a conditional version of Roy's identity, yielding the uncompensated demands for goods 1 and 2 conditional upon the discrete choice between goods 1 and 2:

$$-\frac{\partial v^*(\mathbf{p}, y)/\partial p_1}{\partial v^*(\mathbf{p}, y)/\partial y} = \begin{cases} -\dfrac{\partial \tilde{v}_1(p_1, y)/\partial p_1}{\partial \tilde{v}_1(p_1, y)/\partial y} = \tilde{x}_1 & \text{if } k = 1 \\[2ex] -\dfrac{\partial \tilde{v}_2(p_2, y)/\partial p_1}{\partial \tilde{v}_2(p_2, y)/\partial y} = 0 & \text{if } k = 2 \end{cases} \quad (7.3)$$

$$-\frac{\partial v^*(\mathbf{p}, y)/\partial p_2}{\partial v^*(\mathbf{p}, y)/\partial y} = \begin{cases} -\dfrac{\partial \tilde{v}_1(p_1, y)/\partial p_2}{\partial \tilde{v}_1(p_1, y)/\partial y} = 0 & \text{if } k = 1 \\[2ex] -\dfrac{\partial \tilde{v}_2(p_2, y)/\partial p_2}{\partial \tilde{v}_2(p_2, y)/\partial y} = \tilde{x}_2 & \text{if } k = 2 \end{cases}$$

Note that (7.3) defines the conditional marginal utility of income in terms of $y$ rather than $y_{12}$; indeed the presence of the numeraire good introduces no additional complications to the normal workings of Roy's identity. In practical terms, the budget share for the numeraire good can be seen as a flexible quantity, with the ability to expand or contract in line with the residual income after consumption of good 1 or good 2 (see Batley (2012) for further discussion in the context of income and substitution effects).

Although (7.2) and (7.3) usefully introduce the 'primal' problem of utility maximisation, S&R develop most of their analysis (an important exception being their discussion of welfare, reviewed in section 6 of the present chapter) from the 'dual' perspective of minimising the expenditure necessary to achieve utility $u^*$. That is:

$$e(\mathbf{p}, u^*) = \tilde{e}_k(p_k, u^*) = \text{Min}\{\tilde{e}_1(p_1, u^*), \tilde{e}_2(p_2, u^*)\}$$

where $e$ is the (dual) expenditure function corresponding to the (primal) indirect utility function, $\tilde{e}_k$ is the conditional expenditure function, and $k$ again indexes the chosen (i.e. cost minimising) good, i.e. $k = 1$ if $\tilde{e}_1 \leq \tilde{e}_2$, or $k = 2$ otherwise.[3] Corresponding to Roy's identity (7.3), S&R employ a conditional version of Shephard's lemma, yielding the compensated demands for goods 1 and 2 conditional upon the discrete choice between goods 1 and 2:

$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_1} = \begin{cases} \dfrac{\partial \tilde{e}_1(p_1, u^*)}{\partial p_1} = \tilde{x}_1^c & \text{if } k = 1 \\[2ex] \dfrac{\partial \tilde{e}_2(p_2, u^*)}{\partial p_1} = 0 & \text{if } k = 2 \end{cases} \tag{7.4}$$

$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_2} = \begin{cases} \dfrac{\partial \tilde{e}_1(p_1, u^*)}{\partial p_2} = 0 & \text{if } k = 1 \\[2ex] \dfrac{\partial \tilde{e}_2(p_2, u^*)}{\partial p_2} = \tilde{x}_2^c & \text{if } k = 2 \end{cases}$$

where the $c$ superscript distinguishes the conditional compensated demand $\tilde{x}_1^c$ for good 1 in (7.4) from the conditional uncompensated demand $\tilde{x}_1$ for good 1 in (7.3), and similarly for good 2.

## 2.3   Discrete-continuous demand, in practice

S&R reconcile the unconditional and conditional demands via the following constructs (their equation 3.16):

$$\left.\begin{array}{l} x_j(\mathbf{p}, y) = \delta_j(\mathbf{p}, y)\,\tilde{x}_j(p_j, y) \\[1ex] x_j^c(\mathbf{p}, u) = \delta_j^c(\mathbf{p}, u)\,\tilde{x}_j^c(p_j, u) \end{array}\right\} \text{ for } j = 1, 2$$

where $\delta_j$ and $\delta_j^c$ are uncompensated and compensated discrete choice indexes respectively, $\delta_1 = \delta_1^c = 1$ if $k = 1$ (i.e. if $\tilde{v}_1 \geq \tilde{v}_2$, $\tilde{e}_1 \leq \tilde{e}_2$, and good 1 is therefore chosen) or $\delta_1 = \delta_1^c = 0$ if $k = 2$, and $\tilde{x}_j$ and $\tilde{x}_j^c$ are con-

ditional uncompensated and compensated demands respectively. In other words:

$$
\begin{aligned}
\delta_j(\mathbf{p}, y) &= \begin{cases} 1 & \tilde{v}_j(p_j, y) \geq \tilde{v}_m(p_m, y) \\ 0 & \text{otherwise} \end{cases} \\[2mm]
\delta_j^c(\mathbf{p}, u) &= \begin{cases} 1 & \tilde{e}_j(p_j, u) \leq \tilde{e}_m(p_m, u) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\Bigg\} \quad \text{for } j, m = 1, 2 \; m \neq j
\tag{7.5}
$$

Developing the analysis further, S&R translate their model of discrete-continuous demand from the individual level to the aggregate. More specifically, and denoting an individual by the index $i = 1, \ldots, N$, they define uncompensated and compensated aggregate demands, respectively:

$$
\begin{aligned}
X_j &= \sum_{i=1}^{N} x_j^i = \sum_{i=1}^{N} \delta_j^i(\mathbf{p}, y^i) \tilde{x}_j^i(p_j, y^i) \\
X_j^c &= \sum_{i=1}^{N} x_j^{c,i} = \sum_{i=1}^{N} \delta_j^{c,i}(\mathbf{p}, u^i) \tilde{x}_j^{c,i}(p_j, u^i)
\end{aligned}
\Bigg\} \quad \text{for } j = 1, 2
\tag{7.6}
$$

Thus the aggregate uncompensated and compensated demands arise from the summation of the respective individual-level demands over individuals. In aggregating across individuals, the following two identities must hold. First, the number of individuals comprising the population arises from the sum of discrete choice indices, in the following manner:

$$
\sum_{i=1}^{N} \sum_{j=1,2} \delta_j^i(\mathbf{p}, y^i) = \sum_{i=1}^{N} \sum_{j=1,2} \delta_j^{c,i}(\mathbf{p}, u^i) = N
$$

Second, aggregate income for the population arises from the sum of individual-level budgets, thus:

$$
Y = \sum_{i=1}^{N} y^i
$$

Mindful perhaps of the flexibility that it brings when aggregating surpluses across individuals (see the later discussion in section 6), S&R re-couch the aggregate demands (7.6) in terms of a 'representative individual', with uncompensated and compensated demands $\tilde{x}_j$ and $\tilde{x}_j^c$ respectively, and budget $y$ (i.e. omitting the $i$ superscript of the earlier notation). Furthermore, the discrete choice index $\delta_j$ is, for the representative individual, replaced by the discrete choice probability $\pi_j$. Whilst S&R are not explicit about the derivation of the latter, it would seem uncontroversial to

suggest that probability arises from the enumeration of the discrete choice index across the population, i.e.

$$\pi_j = \frac{\sum_{i=1}^{N} \delta_j^i}{N} \quad \text{for } j = 1, 2 \tag{7.7}$$

where $0 \leq \pi_j \leq 1$. Using this apparatus of the representative individual, S&R re-state the aggregate demands (7.6), thus:

$$
\begin{aligned}
X_j &= \sum_{i=1}^{N} x_j^i(\mathbf{p}, y^i) = \sum_{i=1}^{N} (\delta_j^i(\mathbf{p}, y^i) \tilde{x}_j^i(p_j, y^i))^{\langle y^i = y, \forall i \rangle} \\
&= \sum_{i=1}^{N} (\delta_j^i(\mathbf{p}, y) \tilde{x}_j^i(p_j, y)) \\
&= \left( \sum_{i=1}^{N} \delta_j^i(\mathbf{p}, y) \right) \tilde{x}_j(p_j, y) \\
&= N\pi_j(\mathbf{p}, y) \tilde{x}_j(p_j, y) \\
X_j^c &= \sum_{i=1}^{N} x_j^{c,i}(\mathbf{p}, u^i) = \sum_{i=1}^{N} (\delta_j^{c,i}(\mathbf{p}, u^i) \tilde{x}_j^{c,i}(p_j, u^i)) \\
&= N\pi_j^c(\mathbf{p}, \{u^i\}) \tilde{x}_j^c(p_j, \{u^i\})
\end{aligned}
\right\} \quad \text{for } j = 1, 2
\tag{7.8}
$$

and the aggregate budget:

$$Y = Ny \tag{7.9}$$

Note with reference to (7.6) and (7.8) that (7.7) holds only if the conditional uncompensated demand for each individual is the same (hence the notion of the representative individual), i.e. if:

$$\tilde{x}_j^i(p_j, y^i) = \tilde{x}_j(p_j, y) \quad \text{for } i = 1, \ldots, N; \, j = 1, 2$$

since:

$$\pi_j(\mathbf{p}, y) = \frac{\sum_{i=1}^{N} \left( \delta_j^i(\mathbf{p}, y^i) \tilde{x}_j^i(p_j, y^i) \right)}{N \tilde{x}_j(p_j, y)} \quad \text{for } j = 1, 2$$

This is especially true for the case $\tilde{x}_1 = \tilde{x}_2 = 1$, which will be an important focus of the subsequent discussion. Whereas the aggregate uncompensated demand assumes income to be common for all individuals $i = 1, \ldots, N$, note that the aggregate compensated demand depends upon the utilities of all individuals, i.e. $\{u^i\}$.

Having replaced the 0/1 discrete choice index at the individual level with a discrete choice probability at the aggregate level (i.e. applying to the representative individual), we are called upon to revise our interpretation of $X_j$ and $X_j^c$. In particular, it would seem appropriate to interpret $X_j$ and $X_j^c$ from (7.8) as 'expected' demands. This reflects the fact that, if both choice probability and conditional demand are non-zero for both goods 1 and 2 (i.e. $\pi_1\pi_2 > 0$ and $\tilde{x}_1\tilde{x}_2 > 0$), it must be the case that $X_1X_2 > 0$, contrasting with the constraint $x_1x_2 = 0$ applying to (7.1). Note that this distinction becomes especially relevant when considering the prevalence of income effects of price changes (Batley, 2012).

## 2.4 Eliciting a probabilistic demand model

It is worth remarking that the expected demands (7.8) are routinely exploited in practical demand analysis, for example in the transportation sector. A popular assumption (e.g. Jara-Díaz and Farah, 1988, eq. 60) however, is that the conditional demands for both goods 1 and 2 are given by a common constant (i.e. $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, noting that $\tilde{x}_j = \tilde{x}_j^c$ for $j = 1, 2$ will hold in equilibrium). For example, Hau (1985) observes that: 'It is in the nature of discrete travel choice models that the total number of trips is assumed fixed' (p. 480). Of particular interest to the present chapter is the case where $\tilde{x}_1 = \tilde{x}_2 = 1$, which would apply if the analysis were restricted to a single discrete choice between goods 1 and 2. As Jara-Díaz and Farah (1988) (their equations 53–57) recognise, in this case, Roy's identity (7.3) implies that the (absolute) conditional marginal utilities of price and income should be equal, and common to both goods 1 and 2:

$$-\frac{\partial \tilde{v}_1}{\partial p_1} = \frac{\partial \tilde{v}_1}{\partial y} = -\frac{\partial \tilde{v}_2}{\partial p_2} = \frac{\partial \tilde{v}_2}{\partial y}$$

Ibáñez and Batley (2011) derive the same result in two alternative ways, using Lagrangian methods and the envelope theorem (the latter being the more general in that it can straightforwardly accommodate multiple constraints). We shall devote further discussion to this equality between the conditional marginal utilities of price and income when considering issues of model specification in section 4 of the present chapter.

Moreover if, having restricted consumption to a single unit, we accept the functional representation of the discrete choice index in terms of prices, income and utility (7.5), and admit the process of aggregating deterministic individual-level discrete choice indices to yield a probabilistic index for the population (7.7), then we arrive at the notion of a probabilistic demand function. Indeed, Hau (1985, 1987) adopts this perspective at the

outset, deriving the probabilistic demand function from both primal and dual approaches.

## 3.   THE SLUTSKY EQUATION FOR DISCRETE-CONTINUOUS DEMAND

### 3.1   Introduction

As is widely understood and accepted, the Slutsky equation (Slutsky, 1915) embodies a fundamental relationship between individual-level uncompensated (Marshallian) and compensated (Hicksian) demand functions. Within the Hicksian analysis, a price change influences demand simply through a 'substitution effect', whereas within the Marshallian analysis, the change in demand corresponding to a price change can be dissected into both a substitution effect and an 'income effect'. Following S&R, the present chapter will derive the Slutsky equation for the case of discrete-continuous demand, contrasting with the literature's usual focus on the continuous demand.

With reference to the primal problem, S&R specify probabilistic and conditional demands as a function of prices and income in the manner of (7.8). A minor distinction from S&R (specifically their equation 3.24) is that (7.8) specifies demand as functional on the income of the representative individual $y$ rather than aggregate income $Y$ (although, since we assume $Y = Ny$, the dependencies on $Y$ and $y$ are in fact equivalent). Moreover, (7.8) can be seen as a statement of aggregate (expected) demand; this arises from the product of the probabilistic and conditional demands, further multiplied by the number of individuals in the population. S&R assume that, in equilibrium, the uncompensated conditional and probabilistic demands are equivalent to their compensated counterparts, specifically:

$$\tilde{x}_j^c(p_j, \{u^i\}) = \tilde{x}_j(p_j, \tilde{e}_j(p_j, \{u^i\}))  \text{ for } j = 1, 2 \qquad (7.10)$$

$$\pi_j^c(\mathbf{p}, \{u^i\}) = \pi_j(\mathbf{p}, \{e^i(\mathbf{p}, u^i)\})  \quad \text{for } i = 1, \ldots, N; j = 1, 2 \quad (7.11)$$

Note that (7.10) is a function of the expenditure of the representative individual, whereas (7.11) indexes expenditure by individual. The subsequent discussion will proceed by first deriving the Slutsky equation separately for both the conditional demand $\tilde{x}_j$ and the probabilistic demand $\pi_j$, before then deriving the Slutsky equation for the aggregate (expected) demand $X_j$.

### 3.2 Slutsky equation for the conditional demand

From the equilibrium condition for the conditional demand (7.10), we can derive the Slutsky equation (S&R's equation 3.18), by making use of Shephard's lemma (7.4) for the conditional demand, and assuming that $\widetilde{x}_j^c = \widetilde{x}_j$ in equilibrium:[4]

$$\frac{\partial \widetilde{x}_j^c}{\partial p_j} = \frac{\partial \widetilde{x}_j}{\partial p_j} + \frac{\partial \widetilde{x}_j}{\partial y}\widetilde{x}_j \text{ for } j = 1, 2 \tag{7.12}$$

As is widely understood, the first element on the right hand side of (7.12) embodies the substitution effect of a change in price, whilst the second element (i.e. product of two terms) embodies the income effect.

### 3.3 Slutsky equation for the probabilistic demand

Proceeding to the equilibrium condition for the probabilistic demand (7.11), we can derive the analogous Slutsky equation. Thus, taking derivatives on both sides of the identity, applying the chain rule for derivatives, and acknowledging that conditional expenditure is independent of the price of *all* goods apart from the one that the demand refers to, we get the following:

$$\frac{\partial \pi_j^c(\mathbf{p}, \{u^1, \ldots, u^N\})}{\partial p_j} = \frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial p_j}\Bigg|_{\{y^i\} = \{\widetilde{e}_{k_i}^i(p_{k_i}, u^i)\}}$$

$$+ \sum_{i=1}^{N}\left(\frac{\partial \pi_j(\mathbf{p}, \{\widetilde{e}_{k_1}^1(p_{k_1}, u^1), \ldots, \widetilde{e}_{k_N}^N(p_{k_N}, u^N)\})}{\partial \widetilde{e}_{k_i}^i(p_{k_i}, u^i)} \cdot \frac{\partial \widetilde{e}_{k_i}^i(p_{k_i}, u^i)}{\partial p_j}\right)$$

$$= \frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial p_j}\Bigg|_{\{y^i\} = \{\widetilde{e}_{k_i}^i(p_{k_i}, u^i)\}}$$

$$+ \sum_{i=1}^{N}\left(\frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial y^i}\Bigg|_{\{y^i\} = \{\widetilde{e}_{k_i}^i(p_{k_i}, u^i)\}} \cdot \delta_j^{c,i}(\mathbf{p}, u^i) \cdot \widetilde{x}_j^{c,i}(p_j, u^i)\right)$$

$$\text{for } i = 1, \ldots, N; j = 1, 2$$

Digressing slightly, note that the assumption of identical incomes associated with the representative individual ensures that the following result holds in equilibrium:

$$\frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial y^i}\Bigg|_{\{y^i\} = \{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}} = \frac{\partial \pi_j(\mathbf{p}, y)}{\partial y}\Bigg|_{y^i = \{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}} \qquad \text{for } i = 1, \ldots, N; j = 1, 2 \tag{7.13}$$

where $y = \{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}$ is meaningful given that the $N$ elements in $\{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}$ have identical values at equilibrium. By contrast, there is no need to impose identical utility levels for all individuals.

Returning to our derivation of the Slutsky equation, and making use of (7.13) and (7.8):

$$\frac{\partial \pi^c_j(\mathbf{p}, \{u^1, \ldots, u^N\})}{\partial p_j} = \frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial p_j}\Bigg|_{\{y^i\} = \{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}}$$

$$+ \frac{\partial \pi_j(\mathbf{p}, y)}{\partial y}\Bigg|_{y^i = \{\tilde{e}^i_{k_i}(p_{k_i}, u^i)\}} \cdot X^c_j(\mathbf{p}, \{u^i\}) \qquad \text{for } i = 1, \ldots, N; j = 1, 2 \tag{7.14}$$

Equation (7.14) closely resembles S&R's equation 3.25. We can make further progress towards aligning our analysis with S&R if we suppress functional dependence, acknowledge that in equilibrium it must hold that $X^c_j = X_j$ (this follows from (7.10) and (7.11)), and admit the following equality between derivatives taken with respect to individual income and aggregate income:

$$\frac{\partial \pi_j(\mathbf{p}, y)}{\partial y} = \frac{\partial \pi_j(\mathbf{p}, Y)}{\partial Y} \text{ for } j = 1, 2$$

On this basis, we arrive finally at S&R's Slutsky equation for the probabilistic demand:

$$\frac{\partial \pi^c_j}{\partial p_j} = \frac{\partial \pi_j}{\partial p_j} + \frac{\partial \pi_j}{\partial Y} X_j \text{ for } j = 1, 2 \tag{7.15}$$

and in so doing expose the assumptions inherent within S&R's derivation. In a similar fashion to (7.12), we can interpret the two elements on the right-hand side of (7.15) as the substitution effect and income effect respectively.

### 3.4 Slutsky equation for the aggregate demand

Drawing together the conditional and probabilistic demands, we are now equipped to derive the Slutsky equation for the aggregate (expected) demand, as follows:

$$\frac{\partial X_j^c}{\partial p_j} = N\left(\frac{\partial \pi_j^c}{\partial p_j}\widetilde{x}_j^c + \pi_j^c\frac{\partial \widetilde{x}_j^c}{\partial p_j}\right)$$

$$= N\widetilde{x}_j^c\frac{\partial \pi_j^c}{\partial p_j} + N\pi_j^c\left(\frac{\partial \widetilde{x}_j}{\partial p_j} + \widetilde{x}_j\frac{\partial \widetilde{x}_j}{\partial y}\right)$$

$$= \frac{X_j^c}{\pi_j^c}\frac{\partial \pi_j^c}{\partial p_j} + N\pi_j^c\left(\frac{\partial}{\partial p_j}\left(\frac{X_j}{N\pi_j}\right) + N\widetilde{x}_j\frac{\partial}{\partial Y}\left(\frac{X_j}{N\pi_j}\right)\right)$$

$$= \frac{X_j^c}{\pi_j^c}\left(\frac{\partial \pi_j(\mathbf{p}, \{y^i\})}{\partial p_j}\bigg|_{\{y^i\} = \{e^i(\mathbf{p}, u^i)\}} + X_j^c\frac{\partial \pi_j(\mathbf{p}, Y)}{\partial Y}\right)$$

$$+ N\pi_j^c\left(\frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial p_j} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial p_j}\right) + N\frac{X_j}{N\pi_j}\frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial Y} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial Y}\right)\right)$$

$$= \frac{X_j^c}{\pi_j^c}\left(X_j^c\frac{\partial \pi_j}{\partial Y}\right) + N\pi_j^c\left(\frac{1}{N\pi_j}\frac{\partial X_j}{\partial p_j} + N\frac{X_j}{N\pi_j}\frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial Y} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial Y}\right)\right)$$

$$= \frac{\partial X_j}{\partial p_j} + \frac{X_j}{\pi_j}\left(\frac{\partial X_j}{\partial Y} - (1 - \pi_j)\frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial Y}\right)$$

$$\text{for } i = 1,\dots, N; \ j = 1, 2 \tag{7.16}$$

Note that the second equality of (7.16) employs the Slutsky equation for the conditional demand (7.12), that the third equality employs the definitions of aggregate demand (7.8) and income (7.9), and that the fourth equality makes use of both the Slutsky equation for probabilistic demand (7.15) and the identities (7.17).

$$\frac{\partial \widetilde{x}_j}{\partial p_j} = \frac{\partial}{\partial p_j}\left(\frac{X_j}{N\pi_j}\right) = \frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial p_j} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial p_j}\right)$$

$$\frac{\partial \widetilde{x}_j}{\partial y} = \frac{\partial}{\partial y}\left(\frac{X_j}{N\pi_j}\right) = \frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial y} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial y}\right)$$

$$\frac{\partial \widetilde{x}_j}{\partial Y} = \frac{\partial}{\partial Y}\left(\frac{X_j}{N\pi_j}\right) = \frac{1}{N\pi_j}\left(\frac{\partial X_j}{\partial Y} - \frac{X_j}{\pi_j}\frac{\partial \pi_j}{\partial Y}\right)$$

$$\text{for } j = 1, 2 \tag{7.17}$$

The final equality of (7.16) changes variables to consider derivatives with respect to total income rather than individual income. Note that (7.16)

replicates S&R's Slutsky equation for the aggregate demand (their equation 3.26).

# 4. IMPLICATIONS OF THEORY FOR THE EMPIRICAL SPECIFICATION OF DISCRETE CHOICE MODELS

## 4.1 Introduction

An interest which is developed by Hau (1985, 1987), but not by S&R, is the theoretical derivation of the probabilistic demand function from the dual problems of utility maximisation and cost minimisation. The present section will show that this derivation implies particular requirements on model specification.

## 4.2 Econometric specification of the probabilistic demand

In order to support the subsequent discussion, it will be helpful to assign further form to the specification of probabilistic demand, in the following respects.

First, and following convention in discrete choice modelling, let us specify the conditional indirect utility function:

$$\tilde{v}_j^i(p_j, y) = W_j(p_j, y) + \varepsilon_j^i \quad \text{for } i = 1, \ldots, N; j = 1, 2 \qquad (7.18)$$

where $W_j$ is the deterministic utility of good $j$, the form of which is common to all individuals, and $\varepsilon_j^i$ is a random term, which is specific to good $j$ and individual $i$. The above specification is a slightly simplified form of that assumed by S&R (their equation 5.1). Note that deterministic utility is assumed to be functional upon price and the income of the representative consumer, whereas random error is independent of price and income.

Second, let us relate utility to the uncompensated probabilistic demand through the apparatus of the Random Utility Model (RUM) (Marschak, 1960; Block and Marschak, 1960) thus:

$$\begin{aligned}
\pi_j &= \Pr\{\tilde{v}_j \geq \tilde{v}_m\} = \Pr\{W_j + \varepsilon_j \geq W_m + \varepsilon_m\} \\
&= \Pr\{W_j - W_m \geq \varepsilon_m - \varepsilon_j\} \\
&= \varphi(W_j - W_m) \\
&\quad \text{for } j, m = 1, 2 \ \ m \neq j
\end{aligned} \qquad (7.19)$$

where $\varphi$ is the distribution function of $\varepsilon_m - \varepsilon_j$. RUM itself embodies several properties (see Daly and Zachary (1978) for a comprehensive discussion), but one which will be especially relevant to the subsequent discussion is 'translational invariance', meaning that probability is invariant to any (common) increasing linear transformation of $\tilde{v}_j$. More specifically, probability depends on the difference between the deterministic utilities $(W_j - W_m)$, rather than the absolute values of $W_j$ and $W_m$.

Having introduced (7.18) and (7.19), it should be clarified that the present chapter will, in exploring the implications of S&R for the practical specification of probabilistic demand models, focus upon deterministic utility $W_j(.)$ rather than random error $\varepsilon_j$. This is not to overlook important specification issues concerning random error, which have been considered by a number of previous authors; for a recent review and reassessment of those issues, see Ibáñez (2007).

### 4.3   Five assumptions concerning model specification

Building upon earlier discussion in section 2.4, we begin by restricting S&R's problem of discrete-continuous demand (7.1) to the context of a single discrete choice. This restriction will be relevant to our subsequent derivation of consumer surplus from the probabilistic demand in section 6, and calls for the following assumption:

*Assumption I:   unit conditional demand for goods 1 and 2, i.e.* $\tilde{x}_1 = \tilde{x}_2 = 1$

Following the usual conventions of duality, Hau (1985, 1987) derives the probabilistic demand function via both Roy's identity and Shephard's lemma. This exposes four further assumptions, which are concerned with the specification of deterministic utility.

First and foremost, given Assumption I, the outcome of Roy's identity (7.3) for the conditional demand must be $\tilde{x}_1 = \tilde{x}_2 = 1$, which implies:

*Assumption II: for each good, equivalence (in absolute terms) between the conditional marginal utilities of income and price, i.e.* $-\partial W_1/\partial p_1 = \partial W_1/\partial y, -\partial W_2/\partial p_2 = \partial W_2/\partial y$

Now focusing more specifically upon Hau's derivation, the outcome of Roy's identity for the probabilistic demand must be $\pi_j$. This result implies that Assumption II should be combined with:

*Assumption III:   common conditional marginal utility of income across goods, i.e.* $\partial W_1/\partial y = \partial W_2/\partial y = \lambda$

Note furthermore that Assumptions II and III, taken together, imply a further assumption:

*Assumption IV: common conditional marginal utility of price across goods, i.e.* $\partial W_1/\partial p_1 = \partial W_2/\partial p_2 = -\lambda$

Whilst the norm in transport economic practice is to assume $\partial W_1/\partial p_1 = \partial W_2/\partial p_2$, implying a generic parameter for the conditional marginal utility of price, a number of papers could be cited (e.g. Swait and Ben-Akiva, 1987; de Jong et al., 2003; and Hess et al., 2007) that offer empirical support for $\partial W_1/\partial p_1 \neq \partial W_2/\partial p_2$. Finally, Hau's derivation of the probabilistic demand imposes a fifth assumption:

*Assumption V: the conditional marginal utility of income* $\lambda$ *is independent of the prices of goods 1 and 2, i.e.* $\partial \lambda/\partial p_1 = \partial \lambda/\partial p_2 = 0$

Assumption V eliminates income effects of a price change on the probabilistic demand, implying that – generally speaking[5] – the conditional marginal utility of income will be functional upon income, i.e. $\lambda = \lambda(y)$. We shall consider this property in greater detail in the course of our discussion of welfare in section 6.

Moreover, given Roy's identity (7.3), the above assumptions can be seen to be related in the following manner:

$$\text{Assumption I} \leftrightarrow \text{Assumption II}$$

$$(\text{Assumption II} + \text{Assumption III}) \mapsto \text{Assumption IV}$$

$$(\text{Assumption II} + \text{Assumption IV}) \mapsto \text{Assumption III}$$

In other words, Assumptions I to V reduce to three independent assumptions. Note that a simple specification of the conditional indirect utility function (7.18) which complies with Assumptions I to V is the following:

$$\tilde{v}_j = \beta(y - p_j) + \varepsilon_j \quad \text{for } j = 1, 2$$

where $\beta$ is a parameter to be estimated. This is an example of the 'additive income RUM' (AIRUM) form (McFadden, 1981); given translational invariance, the additive specification means that income exerts no influence on the discrete choice between goods 1 and 2. We shall return to AIRUM in the context of our discussion of homogeneity (section 5.4).

# 5. THEORETICAL PROPERTIES OF THE PROBABILISTIC DEMAND

## 5.1 Introduction

As was noted in Section 1, and is widely understood and accepted (see for example Deaton and Muellbauer (1980) for a general discussion), demand functions should observe four properties, namely 'adding up', 'negativity', 'homogeneity' and 'symmetry'. The purpose of the present section is to consider the extent to which the probabilistic demand emanating from S&R complies with these properties.

In the course of this analysis, we shall draw upon the five assumptions documented in the previous section which, as we have already acknowledged, are inherent within the derivation of the probabilistic demand function. Intuition suggests that there should be correspondence between the assumptions imposed on model specification, and the theoretical properties of the resulting demand function. Generally speaking, the four theoretical properties are relevant to individual-level demands, and may not readily translate to the aggregate demand (a notable example being symmetry). However, having adopted the representative individual, the distinction between individual and aggregate level demands becomes – in the case of S&R – somewhat arbitrary. On this basis, we shall begin our analysis from the perspective of the aggregate (expected) demand (7.8), albeit for the case where $N = 1$ (i.e. notionally applying to a single representative individual). Subsequently, we shall impose further assumptions so as to restrict (7.8) to the probabilistic demand.

## 5.2 Adding up

The 'adding up' property requires that aggregate expenditure on goods 1 and 2 – from both Marshallian and Hicksian perspectives – is equal to budget:

$$p_1 x_1(\mathbf{p}, y) + p_2 x_2(\mathbf{p}, y) = p_1 x_1^c(\mathbf{p}, u) + p_2 x_2^c(\mathbf{p}, u) = y$$

Since S&R specify the budget constraint in (7.1) as an equality, all budget is exhausted and adding up is imposed by definition. With reference to our earlier comment in section 2.2, consumption of the numeraire good can expand or contract in line with residual income after the consumption of goods 1 and 2. In this way, total consumption will always just exhaust budget. Moreover, the numeraire good introduces no substantive

complications into our discussion of the four theoretical properties, and the subsequent sections will therefore assume $x_n = 0$ and $y = y_{12}$.

## 5.3   Negativity

As is widely understood and accepted, a fundamental requirement on demand functions is compliance with the so-called 'integrability' conditions (Hurwicz and Uzawa, 1971; Houthakker, 1950; Samuelson, 1950). These conditions require the compensated demand function to exhibit a symmetric negative semi-definite substitution matrix. We shall deal with symmetry in Section 5.5. The present section considers the 'negativity' property, which requires that the partial derivatives of the compensated demands with respect to own price are less than or equal to zero, i.e.

$$\frac{\partial x_j^c}{\partial p_j} \le 0 \ \text{for} \ j = 1, 2$$

Moreover, an implication of negativity is that, for both goods 1 and 2, any (positive) income effect on demand should be outweighed by a (negative) substitution effect. Substituting for the Slutsky equations from the conditional (7.12) and probabilistic (7.15) demands, we can re-state negativity as follows:

$$\frac{\partial x_j^c}{\partial p_j} = \left[ \left\{ \frac{\partial \pi_j}{\partial p_j} + \frac{\partial \pi_j}{\partial y} x_j \right\} \tilde{x}_j^c + \pi_j^c \left\{ \frac{\partial \tilde{x}_j}{\partial p_j} + \frac{\partial \tilde{x}_j}{\partial y} \tilde{x}_j \right\} \right] \le 0 \ \text{for} \ j = 1, 2$$

Then admitting the econometric specification (7.18), we can expand $\partial \pi_j / \partial p_j$ and $\partial \pi_j / \partial y$ thus:

$$\frac{\partial x_j^c}{\partial p_j} = \left[ \left\{ \frac{\partial \pi_j}{\partial W_j} \frac{\partial W_j}{\partial p_j} + \frac{\partial \pi_j}{\partial W_j} \frac{\partial W_j}{\partial y} x_j \right\} \tilde{x}_j^c + \pi_j^c \left\{ \frac{\partial \tilde{x}_j}{\partial p_j} + \frac{\partial \tilde{x}_j}{\partial y} \tilde{x}_j \right\} \right] \le 0 \ \text{for} \ j = 1, 2 \tag{7.20}$$

Given the generality of (7.20), it is difficult on initial inspection to discern insights for the practical specification of the probabilistic demand. However, we can make progress on this matter if we impose Assumption I; this assumes a single unit of conditional demand (i.e. $\tilde{x}_j = \tilde{x}_j^c = 1$) and implies independence of the conditional demand from price and income (i.e. $\partial \tilde{x}_j / \partial p_j = \partial \tilde{x}_j / \partial y = 0$). In this case, the aggregate (expected) demand effectively becomes the probabilistic demand and (7.20) simplifies to:

$$\frac{\partial x_j^c}{\partial p_j} = \left[ \frac{\partial \pi_j}{\partial W_j} \frac{\partial W_j}{\partial p_j} + \frac{\partial \pi_j}{\partial W_j} \frac{\partial W_j}{\partial y} \pi_j \right] \le 0 \ \text{for} \ j = 1, 2 \tag{7.21}$$

Further simplification of (7.21) yields a final statement of the requirements for negativity, thus:

$$-\frac{\partial W_j}{\partial p_j} \geq \frac{\partial W_j}{\partial y}\pi_j \ \text{ for } j = 1, 2 \tag{7.22}$$

Since the restricted case $\tilde{x}_j = \tilde{x}_j^c = 1$ implies that Assumption II must also hold, and $0 \leq \pi_j \leq 1$ by definition, we conclude that negativity is assured.

As a closing remark, it is appropriate to emphasise that the negativity property (7.22) refers to the restricted case of the probabilistic demand. The negativity property (7.20) applying to the general case of the aggregate (expected) demand lends itself less readily to the elicitation of clear prescription for model specification. The same caveat will apply to the discussions of homogeneity and symmetry that follow.

### 5.4 Homogeneity

As is well established in the literature on 'homogeneity', the compensated demands should be homogenous of degree zero in prices, and the uncompensated demands should be homogenous of degree zero in prices and income. With reference to Euler's Theorem, the implication follows that the marginal utility of income should be homogenous of degree minus one in prices. These properties arise from the linear budget constraint in (7.1), and ensure the absence of 'money illusion'. More formally, for a positive factor $(\theta > 0)$, it must in equilibrium hold that:

$$x_j^c(\theta\mathbf{p}, u) = x_j^c(\mathbf{p}, u) = x_j(\theta\mathbf{p}, \theta y) = x_j(\mathbf{p}, y) \ \ \text{for } j = 1, 2 \tag{7.23}$$

That is to say, if all prices and income are increased by a common factor then demand is unchanged.

An alternative formalisation of homogeneity is in terms of the following identity (where for brevity we restrict attention to the uncompensated demands, although a similar identity could be formulated in respect of the compensated demands):

$$\frac{\partial x_j}{\partial p_1}p_1 + \frac{\partial x_j}{\partial p_2}p_2 + \frac{\partial x_j}{\partial y}y = 0 \ \text{ for } j = 1, 2 \tag{7.24}$$

For a derivation of (7.24) from (7.23), see for example DeSerpa (1971), although note the typographical error in his equation 2.16, which is corrected above. With reference to the aggregate demands (7.8), and also

drawing upon (7.18) and (7.19), let us derive the various components of (7.24), thus:

$$\frac{\partial x_j}{\partial p_1} = \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial p_1} \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial p_1}$$

$$\frac{\partial x_j}{\partial p_2} = \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial p_2} \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial p_2}$$

$$\frac{\partial x_j}{\partial y} = \left[ \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial y} + \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial y} \right] \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial y}$$

$$\text{for } j = 1, 2$$

Substituting back into (7.24), we have:

$$\left[ \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial p_1} \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial p_1} \right] p_1 + \left[ \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial p_2} \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial p_2} \right] p_2 +$$

$$\left[ \left\{ \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial y} + \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial y} \right\} \widetilde{x}_j + \pi_j \frac{\partial \widetilde{x}_j}{\partial y} \right] y = 0 \text{ for } j = 1, 2 \quad (7.25)$$

Following the same rationale as our discussion of negativity (section 5.3), let us again restrict attention to the probabilistic demand by imposing Assumption I, which permits simplification of (7.25) thus:

$$\left[ \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial p_1} \right] p_1 + \left[ \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial p_2} \right] p_2 + \left[ \frac{\partial \pi_j}{\partial W_1} \frac{\partial W_1}{\partial y} + \frac{\partial \pi_j}{\partial W_2} \frac{\partial W_2}{\partial y} \right] y = 0 \text{ for } j = 1, 2 \quad (7.26)$$

As is widely understood (e.g. Steenburgh, 2008), a property of RUM is that each of the rows and columns of the matrix (7.27) sums to zero:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{W}} = \begin{bmatrix} \dfrac{\partial \pi_1}{\partial W_1} & -\dfrac{\partial \pi_1}{\partial W_2} \\[2ex] -\dfrac{\partial \pi_2}{\partial W_1} & \dfrac{\partial \pi_2}{\partial W_2} \end{bmatrix} \quad (7.27)$$

On this basis, we can further simplify (7.26), thus:

$$\frac{\partial W_1}{\partial p_1} p_1 - \frac{\partial W_2}{\partial p_2} p_2 + \left[ \frac{\partial W_1}{\partial y} - \frac{\partial W_2}{\partial y} \right] y = 0 \quad (7.28)$$

If we impose Assumption II (which is already implied by Assumption I) and Assumption III on (7.28), then homogeneity will be assured, but implicit in this result will be the property $p_1 = p_2$, meaning that goods 1 and 2 have common prices.[6] The property $p_1 = p_2$ is consistent with the probabilistic demand embodying homothetic preferences and, in the absence of any non-price influences on demand (arising through qualitative attributes within deterministic utility, or through the random term (7.18)), implies that $\pi_1 = \pi_2 = 0.5$. It is interesting to relate the above findings to Hau's (1985) assertion (which he does not formally justify) that AIRUM does not comply with homogeneity. AIRUM would arise if Assumptions I, II and III were imposed on RUM, and income entered the conditional indirect utility function additively. Our findings give us cause to qualify Hau's assertion; from (7.28) we can see that AIRUM will comply with homogeneity where $p_1 = p_2$, but will not comply otherwise.

### 5.5 Symmetry

Further to the discussion of negativity in section 5.3, a second feature of the integrability conditions is the 'Slutsky symmetry' property (Slutsky, 1915). Before proceeding, it should be qualified that symmetry is, strictly speaking, relevant to the individual-level demand. Although there is no analogous requirement on the aggregate demand, S&R's adoption of the representative individual implies that their demand function will embody consistent properties, whether viewed in aggregate or disaggregate.

Arising from Shephard's lemma and Young's theorem, the Slutsky symmetry property imposes the following condition on the individual-level unconditional compensated demands:

$$\frac{\partial x_1^c}{\partial p_2} = \frac{\partial}{\partial p_2}\left(\frac{\partial e}{\partial p_1}\right) = \frac{\partial}{\partial p_1}\left(\frac{\partial e}{\partial p_2}\right) = \frac{\partial x_2^c}{\partial p_1}$$

Substituting for the unconditional compensated demands, using our notion of expected demand (7.8), and again restricting attention to the case $N = 1$, the symmetry property can be restated:

$$\frac{\partial(\pi_1^c \tilde{x}_1^c)}{\partial p_2} = \frac{\partial(\pi_2^c \tilde{x}_2^c)}{\partial p_1} \tag{7.29}$$

If, by analogy to (7.12) and (7.15), we derive Slutsky decompositions of the conditional and probabilistic demands in terms of cross-price effects, and draw upon the econometric specification (7.18) and (7.19), then we can expand the terms of (7.29) as follows:

$$\left[\frac{\partial\pi_1}{\partial W_2}\left\{\frac{\partial W_2}{\partial p_2}+\frac{\partial W_2}{\partial y}x_2\right\}\right]\tilde{x}_1^c+\pi_1^c\left[\frac{\partial\tilde{x}_1}{\partial p_2}+\frac{\partial\tilde{x}_1}{\partial y}\tilde{x}_2\right]$$

$$=\left[\frac{\partial\pi_2}{\partial W_1}\left\{\frac{\partial W_1}{\partial p_1}+\frac{\partial W_1}{\partial y}x_1\right\}\right]\tilde{x}_2^c+\pi_2^c\left[\frac{\partial\tilde{x}_2}{\partial p_1}+\frac{\partial\tilde{x}_2}{\partial y}\tilde{x}_1\right] \qquad (7.30)$$

In this way, we derive a statement of Slutsky symmetry, which must hold if the uncompensated demands for goods 1 and 2 are to satisfy integrability.

Now adopting the same focus as the earlier discussions of negativity and homogeneity, let us restrict (7.30) to the probabilistic demand by invoking Assumption I. If we further impose Assumption II (which is implied by Assumption I) and note that $\pi_j = \pi_j^c$ holds in equilibrium, then symmetry amounts to the following equality:

$$[1+\pi_2]\left[\frac{\partial\pi_1}{\partial W_2}\frac{\partial W_2}{\partial p_2}\right]=[1+\pi_1]\left[\frac{\partial\pi_2}{\partial W_1}\frac{\partial W_1}{\partial p_1}\right] \qquad (7.31)$$

In the extant literature, various authors (including S&R themselves and Daly and Zachary (1978)) represent symmetry simply in terms of the cross-partial derivatives of probability with respect to deterministic utility (referred to as 'Condition 5' in Daly and Zachary):

$$\frac{\partial\pi_1}{\partial W_2}=\frac{\partial\pi_2}{\partial W_1} \qquad (7.32)$$

Although necessary to ensure that choice probabilities can be generated by a correctly specified multivariate distribution for the random terms, (7.31) shows that Condition 5 is not in itself sufficient to ensure Slutsky symmetry. Indeed, equation (7.31) reveals that Slutsky symmetry is, in principle, dependent upon not only the cross-partial derivatives of probability with respect to deterministic utility, but also the conditional marginal utilities of price for goods 1 and 2, and the market shares. Moreover, if Assumption IV is imposed and $\pi_1 = \pi_2 = 0.5$ (noting that the latter is implied by homogeneity), then symmetry will hold.

An alternative line of reasoning would be to impose Assumption V on (7.30), which would eliminate income effects of price changes at the outset. If we further impose Assumptions I, II and IV then we again arrive at (7.32). This line of reasoning might be seen as a probabilistic version of the conventional result that a Marshallian demand will, in the absence of income effects of a price change, comply with the symmetry property. Given Assumption IV and common prices, it must again be the case (in the absence of non-price influences on demand) that $\pi_1 = \pi_2 = 0.5$.

# 6. WELFARE MEASURES FROM DISCRETE-CONTINUOUS DEMANDS

## 6.1 Introduction

S&R's paper culminates in the derivation – from their model of discrete-continuous demand – of a consumer surplus measure defined entirely in terms of the probabilistic demand. The following discussion will review this derivation, with the aim of exposing S&R's underlying rationale, and reconciling this rationale with the conclusions from Sections 4 and 5 of the present chapter.

## 6.2 Definitions of Hicksian consumer surplus

S&R begin their discussion of welfare at the individual level, deriving the compensating variation of a change in the price of good $m$ from $p_m^0$ to $p_m^f$ as the integral of the unconditional compensated demand:[7]

$$\Delta e_j = \int_{p_m^0}^{p_m^f} (x_j^{c,i}(\mathbf{p}, u^{0i})) \, dp_m \text{ for } i = 1,\dots, N; \ j, m = 1, 2$$

where the superscript 0 refers to the base case, and $f$ to the forecast case.

Taking the population of consumers as a whole, S&R acknowledge that, whilst facing a common set of prices, different consumers may make different discrete choices and derive different utilities. On this basis, S&R propose an analogue to the compensating variation, defined at the aggregate level, and in terms of the discrete-continuous demand:

$$\Delta E_j = \int_{p_m^0}^{p_m^f} \left( \sum_{i=1}^{N} (\delta_j^{c,i}(\mathbf{p}, u^{0i}) \tilde{x}_j^{c,i}(p_j, u^{0i})) \right) dp_m \text{ for } j, m = 1, 2 \quad (7.33)$$

Alternatively, if we adopt the perspective of the representative individual then, following (7.7), we can replace the deterministic discrete choice indicator in (7.33) with the probabilistic, thus:

$$\Delta E_j = \int_{p_m^0}^{p_m^f} (N\pi_j^c(\mathbf{p}, u^0) \tilde{x}_j^c(p_j, u^0)) \, dp_m \text{ for } j, m = 1, 2$$

## 6.3 Isolating consumer surplus specific to the probabilistic demand

Although S&R introduce the concept of consumer surplus in terms of the compensated (Hicksian) demand, in practice they adopt the uncompensated (Marshallian) demand as a working approximation; we shall

rehearse the rationale behind this approximation in section 6.4. The present section will discuss more general issues concerning the structure of the demand function underpinning S&R's consumer surplus measure.

In deriving consumer surplus, S&R integrate over the product of the probabilistic and conditional (Marshallian) demands, implying that (in general terms) the relevant demand function for welfare measurement is the expected demand:[8]

$$\Delta E_j \approx -\int_{p_m^0}^{p_m^f}\left( N\pi_j(\mathbf{p}, y^0)\frac{\partial W_j/\partial p_j}{\partial W_j/\partial y}\right)dp_m \quad \text{for } j, m = 1, 2 \qquad (7.34)$$

where we replace $e$ with $y$, which must hold in equilibrium, and substitute for the conditional demand using the econometric specification (7.18) and Roy's identity (7.3). S&R then restrict their focus to the probabilistic demand; with reference to (7.18) and (7.19) they observe that: '. . .$\pi_1$ depends on its arguments only through the functions $W_1$ and $W_2$' (p. 124). This observation is used to justify a change of variables, and a re-working of (7.34) which arrives at (essentially) the following expression:

$$\Delta E \approx -\frac{N}{\lambda(y)}\int_{W_j^0}^{W_j^f}(\pi_j(W_j(p_j, y) - W_m(p_m, y)))dW_j \quad \text{for } j, m = 1, 2 \; m \neq j \qquad (7.35)$$

Thus, S&R derive a measure of welfare change by integrating the probabilistic demand for the change in indirect deterministic utility associated with a price change, and converting into money units. The 'log sum' construct used widely in public policy analysis is a special case of (7.35) where the random term in (7.18) is distributed Multivariate Extreme Value (MEV). The translational invariance property means that, in the binary case, (7.35) would give the same answer whether couched in terms of good 1 or good 2.

Having articulated S&R's consumer surplus measure, let us complete this section by rationalising its basis in terms of Assumptions I to V, thus:

- Assumption I restricts (7.1) to a single discrete choice between goods 1 and 2, supporting the transition from (7.34) to (7.35).
- Assumptions II, III, IV and V are inherent within the derivation of the probabilistic demand, which is the focus of the consumer surplus measure (7.35). Assumption II is implied by Assumption I, and ensures that the probabilistic demand complies with the negativity property. Assumptions III, IV and V ensure compliance with the homogeneity and symmetry properties.

### 6.4 Marshallian consumer surplus as an approximation to the Hicksian

S&R introduce three assumptions on the basis of 'purely empirical considerations' (their terminology), two of which are used to justify S&R's adoption of Marshallian consumer surplus as an approximation to Hicksian (as illustrated in (7.34) and (7.35) above). In other words, S&R's consumer surplus measure is applicable only to empirical contexts where these assumptions hold. The first of these two assumptions duplicates Assumption V concerning path independence, whilst the second states that: '. . .the discrete goods are sufficiently unimportant to the consumer so that income effects. . .are negligible, i.e. that the compensated demand. . .is adequately approximated by the Marshallian demand function' (p. 124).

Let us consider the theoretical rationale for each of these assumptions in more detail. Assumption V is motivated by a wish to nullify the path dependency of Marshallian consumer surplus to price changes. This assumption implies that, if homogeneity is to hold, then $\lambda = \lambda(y)$ (see Batley and Ibañez (2012) for a fuller explanation). It follows that if prices change with income fixed then the compensating variation will be proportional to the Marshallian consumer surplus (Samuelson, 1942). This property corroborates our conclusion on symmetry (section 5.5), and implies clear prescription for model specification; if practitioners wish to adhere to S&R, and in particular apply consumer surplus measures in the form of (7.35), then models should be specified such that probability is invariant to any change in real income associated with a price change.

Turning to S&R's second 'empirical' assumption, it is important to note that, whilst Assumption V is sufficient to ensure path independence with respect to price changes in the case of fixed income, the path independence conditions become more involved if we also admit lump sum income changes (Batley and Ibañez, 2012). However, as was revealed in our discussion of homogeneity (section 5.4), the imposition of Assumptions I, II and III implies $p_1 = p_2$ and, in the absence of non-price influences on demand, $\pi_1 = \pi_2 = 0.5$. In other words, preferences are homothetic (giving rise to a linear income expansion path), but conditional demand is fixed at a single unit (restricting any movement along that income expansion path). This is the basis for S&R's assumption that income effects are 'negligible' in the empirical context of application.

## 7. SUMMARY AND CONCLUSIONS

S&R consider a demand problem where commodities are available in continuous quantities but only a limited number of varieties. In particular,

they consider a simple case involving ostensibly two varieties, referred to as goods 1 and 2, together with a third (numeraire) good. S&R distinguish between two components of demand, discrete choice between goods 1 and 2, and consumption of a continuous quantity conditional upon the discrete choice. Emanating from this problem, S&R derive an expected demand function, given by the product of a probabilistic demand function (appealing to the discrete choice between goods 1 and 2) and a conditional demand function (appealing to the continuous quantity of the chosen good). The principal outcome of S&R's analysis is the derivation of a consumer surplus measure specific to the probabilistic demand function. This measure readily lends itself to practical implementation via RUM. Depending upon the distributional assumptions applied to the random term within RUM, the consumer surplus measure may take on different forms. A popular such form is the so-called 'log sum' metric, which arises where the random term is distributed Multivariate Extreme Value (MEV).

The contributions of our own paper were to expose several assumptions that underpin S&R's consumer surplus measure, to consider the implications of these assumptions for the empirical specification of the probabilistic demand, and to clarify the applicability of the resulting model for practical welfare measurement.

More specifically, we exposed five assumptions that underpin S&R, as follows:

*Assumption I:   unit conditional demand for goods 1 and 2, i.e.* $\tilde{x}_1 = \tilde{x}_2 = 1$

*Assumption II: for each good, equivalence (in absolute terms) between the conditional marginal utilities of income and price, i.e.* $-\partial W_1/\partial p_1 = \partial W_1/\partial y, -\partial W_2/\partial p_2 = \partial W_2/\partial y$

*Assumption III:  common conditional marginal utility of income across goods, i.e.* $\partial W_1/\partial y = \partial W_2/\partial y = \lambda$

*Assumption IV:  common conditional marginal utility of price across goods, i.e.* $\partial W_1/\partial p_1 = \partial W_2/\partial p_2 = -\lambda$

*Assumption V:   the conditional marginal utility of income* $\lambda$ *is independent of the prices of goods 1 and 2, i.e.* $\partial \lambda/\partial p_1 = \partial \lambda/\partial p_2 = 0$

Assumption I restricts S&R's consumer surplus measure to the context of a single discrete choice between goods 1 and 2. The remaining assumptions are associated with the derivation of a probabilistic demand function that is valid in terms of the fundamental properties of demand functions.

Assumption II follows from Assumption I, and ensures compliance with negativity, whilst Assumptions III, IV and V ensure compliance with homogeneity and symmetry.

Moreover, the consumer surplus measure emanating from S&R is applicable only to a very particular empirical context. This is the context of discrete choice from a set of mutually exclusive and exhaustive goods, where the goods face common prices, are perfectly substitutable, and income effects of price and income changes are zero. If income effects of price and/or income changes are relevant to the empirical context of application, then Assumptions I to V should be relaxed, and measurement of consumer surplus should be taken from the discrete-continuous demand rather than from the probabilistic demand specifically.

## NOTES

1. In this way, the precise definition of the 'good' is intimately related to the precise definition of the 'budget'. We shall develop this point further in due course.
2. Furthermore, let us arbitrarily assign $k = 1$ to the case $\tilde{v}_1 = \tilde{v}_2$, so as to avoid ambiguities in the event of ties.
3. To be consistent with (7.3), let us arbitrarily assign $k = 1$ to the case $\tilde{e}_1 = \tilde{e}_2$.
4. For notational brevity, the remainder of the chapter will suppress functional dependencies where this is convenient and does not impinge upon the clarity of the analysis.
5. AIRUM would be an exception to the rule; see Batley and Ibáñez (2012) for further discussion in the context of the path independence conditions.
6. With reference to note 1, this property follows from the manner in which the 'goods' and the 'budget' are defined, and is not as restrictive as it might seem. Consider for example a choice between a vacation, with an actual price of £2000, and a 'staycation' with an actual price of £0; let us assume that consumption of the vacation exhausts the available budget. Since the foregoing of the vacation will – in effect – release £2000 for consumption of the numeraire good, we can conceptualise 'good 1' as the vacation (at a unit price of £2000) and 'good 2' as the numeraire consumption associated with the staycation (also at a *notional* unit price of £2000). On this basis, one unit of either good will exhaust the budget.
7. Since a given price change could have welfare implications for both goods, it is appropriate to calculate the surplus for each good separately, and then to aggregate.
8. Once we defer to the Marshallian demands, any aggregation over goods 1 and 2 will in general be subject to path dependence.

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